Lecture 2

Maxwell's Equations, Differential Operator Form

2.1 Gauss's Divergence Theorem

The divergence theorem is one of the most important theorems in vector calculus [30–33]. First, we will need to prove Gauss's divergence theorem, namely, that:

$$\iiint_V dV \nabla \cdot \mathbf{D} = \oiint_S \mathbf{D} \cdot d\mathbf{S} \tag{2.1.1}$$

In the above, $\nabla \cdot \mathbf{D}$ is defined as

$$\nabla \cdot \mathbf{D} = \lim_{\Delta V \to 0} \frac{\oiint_{\Delta S} \mathbf{D} \cdot d\mathbf{S}}{\Delta V} \tag{2.1.2}$$

The above implies that the divergence of the electric flux $\mathbf{D}$, or $\nabla \cdot \mathbf{D}$ is given by first computing the flux coming (or oozing) out of a small volume ΔV surrounded by a small surface ΔS and taking their ratio as shown on the right-hand side. As shall be shown, the ratio has a limit and eventually, we will find an expression for it. We know that if $\Delta V \approx 0$ or small, then the above,

$$\Delta V \nabla \cdot \mathbf{D} \approx \oiint_{\Delta S} \mathbf{D} \cdot d\mathbf{S} \tag{2.1.3}$$

First, we assume that a volume V has been discretized[1] into a sum of small cuboids, where the i-th cuboid has a volume of ΔV_i as shown in Figure 2.1. Then

$$V \approx \sum_{i=1}^{N} \Delta V_i \tag{2.1.4}$$

[1]Other terms are "tesselated", "meshed", or "gridded".

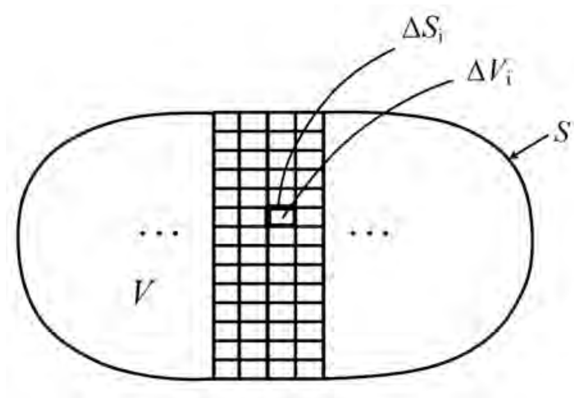

Figure 2.1: The discretization of a volume V into sum of small volumes ΔV_i each of which is a small cuboid. Stair-casing error occurs near the boundary of the volume V but the error diminishes as $\Delta V_i \to 0$.

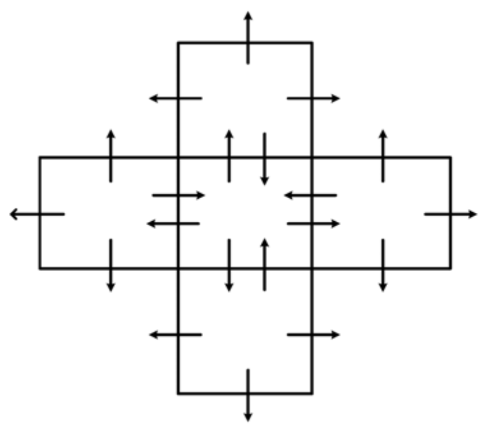

Figure 2.2: Fluxes from adjacent cuboids cancel each other leaving only the fluxes at the boundary that remain uncancelled. Please imagine that there is a third dimension of the cuboids in this picture where it comes out of the paper.

Then from (2.1.2),

$$\Delta V_i \nabla \cdot \mathbf{D}_i \approx \oiint_{\Delta S_i} \mathbf{D}_i \cdot d\mathbf{S}_i \tag{2.1.5}$$

By summing the above over all the cuboids, or over i, one gets

$$\sum_i \Delta V_i \nabla \cdot \mathbf{D}_i \approx \sum_i \oiint_{\Delta S_i} \mathbf{D}_i \cdot d\mathbf{S}_i \approx \oiint_S \mathbf{D} \cdot d\mathbf{S} \tag{2.1.6}$$

It is easily seen that the fluxes out of the inner surfaces of the cuboids cancel each other, leaving only fluxes flowing out of the cuboids at the edge of the volume V as explained in Figure 2.2. The right-hand side of the above equation (2.1.6) becomes a surface integral over the surface S except for the stair-casing approximation (see Figure 2.1). However, this approximation becomes increasingly good as $\Delta V_i \to 0$. Moreover, the left-hand side becomes a volume integral, and we have

$$\iiint_V dV \nabla \cdot \mathbf{D} = \oiint_S \mathbf{D} \cdot d\mathbf{S} \tag{2.1.7}$$

The above is Gauss's divergence theorem.

Next, we will derive the details of the definition embodied in (2.1.2). To this end, we evaluate the numerator of the right-hand side carefully, in accordance to Figure 2.3.

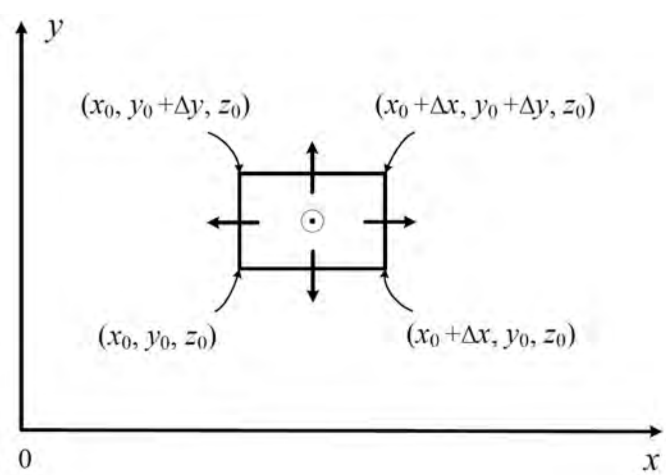

Figure 2.3: Figure to illustrate the calculation of fluxes from a small cuboid where a corner of the cuboid is located at (x_0, y_0, z_0). There is a third z dimension of the cuboid not shown, and coming out of the paper. Hence, this cuboid, unlike that shown in the figure, has six faces.

Accounting for the fluxes going through all the six faces, assigning the appropriate signs in accordance with the fluxes leaving and entering the cuboid, one arrives at

$$\begin{aligned}\oiint_{\Delta S} \mathbf{D} \cdot d\mathbf{S} \approx & -\mathrm{D}_x(x_0, y_0, z_0)\Delta y \Delta z + \mathrm{D}_x(x_0 + \Delta x, y_0, z_0)\Delta y \Delta z \\ & -\mathrm{D}_y(x_0, y_0, z_0)\Delta x \Delta z + \mathrm{D}_y(x_0, y_0 + \Delta y, z_0)\Delta x \Delta z \\ & -\mathrm{D}_z(x_0, y_0, z_0)\Delta x \Delta y + \mathrm{D}_z(x_0, y_0, z_0 + \Delta z)\Delta x \Delta y \end{aligned} \tag{2.1.8}$$

Factoring out the volume of the cuboid $\Delta V = \Delta x \Delta y \Delta z$ in the above, one gets

$$\begin{aligned}\oiint_{\Delta S} \mathbf{D} \cdot d\mathbf{S} \approx \Delta V \{ & \left[D_x(x_0 + \Delta x, \ldots) - D_x(x_0, \ldots) \right] / \Delta x \\ & + \left[D_y(\ldots, y_0 + \Delta y, \ldots) - D_y(\ldots, y_0, \ldots) \right] / \Delta y \\ & + \left[D_z(\ldots, z_0 + \Delta z) - D_z(\ldots, z_0) \right] / \Delta z \} \end{aligned} \tag{2.1.9}$$

Or that

$$\frac{\oiint \mathbf{D} \cdot d\mathbf{S}}{\Delta V} \approx \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \tag{2.1.10}$$

In the limit when $\Delta V \to 0$, then

$$\lim_{\Delta V \to 0} \frac{\oiint \mathbf{D} \cdot d\mathbf{S}}{\Delta V} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \nabla \cdot \mathbf{D} \tag{2.1.11}$$

where

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{2.1.12}$$

$$\mathbf{D} = \hat{x}D_x + \hat{y}D_y + \hat{z}D_z \tag{2.1.13}$$

The above is the definition of the divergence operator in Cartesian coordinates. The divergence operator $\nabla \cdot$ has its complicated representations in cylindrical and spherical coordinates, a subject that we would not delve into in this course. But they are best looked up at the back of some textbooks on electromagnetics.

Consequently, one gets Gauss's divergence theorem given by

$$\iiint_V dV \nabla \cdot \mathbf{D} = \oiint_S \mathbf{D} \cdot d\mathbf{S} \tag{2.1.14}$$

2.1.1 Gauss's Law in Differential Operator Form

By further using Gauss's or Coulomb's law implies that

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = Q = \iiint dV \varrho \tag{2.1.15}$$

We can esplace the left-hand side of the above by (2.1.14) to arrive at

$$\iiint_V dV \nabla \cdot \mathbf{D} = \iiint_V dV \varrho \tag{2.1.16}$$

When $V \to 0$, we arrive at the pointwise relationship, a relationship at a point in space:

$$\nabla \cdot \mathbf{D} = \varrho \tag{2.1.17}$$

2.1.2 Physical Meaning of Divergence Operator

The physical meaning of divergence is that if $\nabla \cdot \mathbf{D} \neq 0$ at a point in space, it implies that there are fluxes oozing or exuding from that point in space [34]. On the other hand, if $\nabla \cdot \mathbf{D} = 0$, if implies no flux oozing out from that point in space. In other words, whatever flux that goes into the point must come out of it. The flux is termed divergence free. Thus, $\nabla \cdot \mathbf{D}$ is a measure of how much sources or sinks exists for the flux at a point. The sum of these sources or sinks gives the amount of flux leaving or entering the surface that surrounds the sources or sinks.

Moreover, if one were to integrate a divergence-free flux over a volume V, and invoking Gauss's divergence theorem, one gets

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = 0 \tag{2.1.18}$$

In such a scenerio, whatever flux that enters the surface S must leave it. In other words, what comes in must go out of the volume V, or that flux is conserved. This is true of incompressible fluid flow, electric flux flow in a source free region, as well as magnetic flux flow, where the flux is conserved.

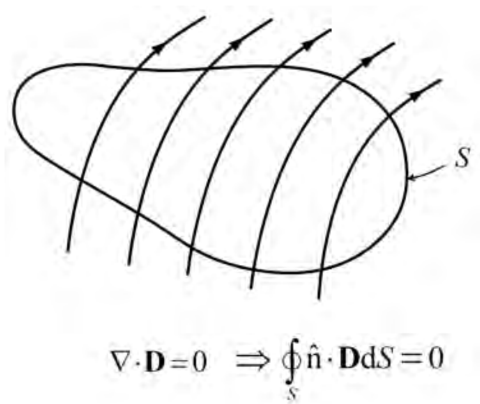

Figure 2.4: In an incompressible flux flow, flux is conserved: whatever flux that enters a volume V must leave the volume V.

2.2 Stokes's Theorem

The mathematical description of fluid flow was well established before the establishment of electromagnetic theory [35]. Hence, much mathematical description of electromagnetic theory uses the language of fluid. In mathematical notations, Stokes's theorem is

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{S} \tag{2.2.1}$$

In the above, the contour C is a closed contour, whereas the surface S is not closed.[2]

First, applying Stokes's theorem to a small surface ΔS, we define a curl operator[3] $\nabla\times$ at a point to be

$$\nabla \times \mathbf{E} \cdot \hat{n} = \lim_{\Delta S \to 0} \frac{\oint_{\Delta C} \mathbf{E} \cdot d\mathbf{l}}{\Delta S} \tag{2.2.2}$$

Taking $\oint_{\Delta C} \mathbf{E} \cdot d\mathbf{l}$ as a measure of the rotation of the field $\mathbf{E}$ around a small loop ΔC, the ratio of this rotation to the area of the loop ΔS has a limit when ΔS becomes infinitesimally small. This ratio is related to $\nabla \times \mathbf{E}$.

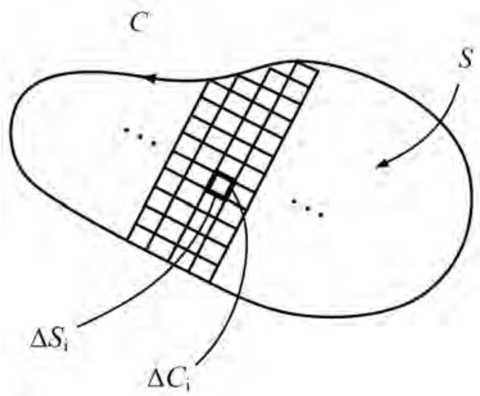

Figure 2.5: In proving Stokes's theorem, a closed contour C is assumed to enclose an open surface S. Then the surface S is tessellated into sum of small rects as shown. Stair-casing error vanishes in the limit when the rects are made vanishingly small.

First, the surface S enclosed by C is tessellated (also called meshed, gridded, or discretized) into sum of small rects (rectangles) as shown in Figure 2.5. Stokes's theorem is then applied to one of these small rects to arrive at

$$\oint_{\Delta C_i} \mathbf{E}_i \cdot d\mathbf{l}_i = (\nabla \times \mathbf{E}_i) \cdot \Delta \mathbf{S}_i \tag{2.2.3}$$

where one defines $\Delta \mathbf{S}_i = \hat{n} \cdot \Delta S$. Next, we sum the above equation over i or over all the small rects to arrive at

$$\sum_i \oint_{\Delta C_i} \mathbf{E}_i \cdot d\mathbf{l}_i = \sum_i \nabla \times \mathbf{E}_i \cdot \Delta \mathbf{S}_i \tag{2.2.4}$$

[2]In other words, C has no boundary whereas S has boundary. A closed surface S has no boundary like when we were proving Gauss's divergence theorem previously.

[3]Sometimes called a rotation operator.

Again, on the left-hand side of the above, all the contour integrals over the small rects cancel each other internal to S save for those on the boundary. In the limit when $\Delta S_i \to 0$, the left-hand side becomes a contour integral over the larger contour C, and the right-hand side becomes a surface integral over S. One arrives at Stokes's theorem, which is

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} \tag{2.2.5}$$

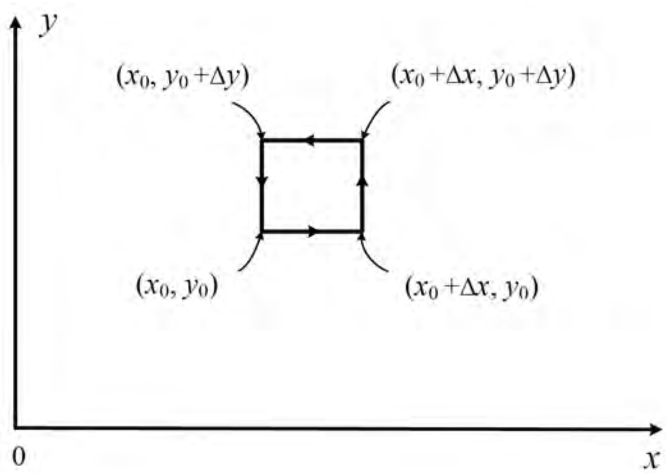

Figure 2.6: We approximate the integration over a small rect using this figure. There are four edges to this small rect.

Next, we need to prove the details of definition (2.2.2) using Figure 2.6. Performing the integral over the small rect, one gets

$$\begin{aligned}\oint_{\Delta C} \mathbf{E} \cdot d\mathbf{l} &= E_x(x_0, y_0, z_0)\Delta x + E_y(x_0 + \Delta x, y_0, z_0)\Delta y \\ &\qquad - E_x(x_0, y_0 + \Delta y, z_0)\Delta x - E_y(x_0, y_0, z_0)\Delta y \\ &= \Delta x \Delta y \left(\frac{E_x(x_0, y_0, z_0)}{\Delta y} - \frac{E_x(x_0, y_0 + \Delta y, z_0)}{\Delta y} \right. \\ &\qquad \left. - \frac{E_y(x_0, y_0, z_0)}{\Delta x} + \frac{E_y(x_0, y_0 + \Delta y, z_0)}{\Delta x} \right)\end{aligned} \tag{2.2.6}$$

We have picked the normal to the incremental surface ΔS to be $\hat{z}$ in the above example, and hence, the above gives rise to the identity that

$$\lim_{\Delta S \to 0} \frac{\oint_{\Delta S} \mathbf{E} \cdot d\mathbf{l}}{\Delta S} = \frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x = \hat{z} \cdot \nabla \times \mathbf{E} \tag{2.2.7}$$

Picking different $\Delta\mathbf{S}$ with different orientations and normals $\hat{n}$ where $|hatn = \hat{x}$ or $\hat{n} = \hat{y}$, one gets

$$\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y = \hat{x}\cdot\nabla\times\mathbf{E} \tag{2.2.8}$$

$$\frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z = \hat{y}\cdot\nabla\times\mathbf{E} \tag{2.2.9}$$

The above gives the x, y, and z components of $\nabla\times\mathbf{E}$. It is to be noted that $\nabla\times\mathbf{E}$ is a vector. Consequently, one gets

$$\nabla\times\mathbf{E} = \hat{x}\left(\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y\right) + \hat{y}\left(\frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z\right) + \hat{z}\left(\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x\right) \tag{2.2.10}$$

where

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{2.2.11}$$

2.2.1 Faraday's Law in Differential Operator Form

Faraday's law is experimentally motivated. Michael Faraday (1791-1867) was an extraordinary experimentalist who documented this law with meticulous care. It was only decades later that a mathematical description of this law was arrived at.

Faraday's law in integral form is given by[4]

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\iint_S \mathbf{B}\cdot d\mathbf{S} \tag{2.2.12}$$

Assuming that the surface S is not time varying, one can take the time derivative into the integrand and write the above as

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\iint_S \frac{\partial}{\partial t}\mathbf{B}\cdot d\mathbf{S} \tag{2.2.13}$$

One can replace the left-hand side with the use of Stokes' theorem to arrive at

$$\iint_S \nabla\times\mathbf{E}\cdot d\mathbf{S} = -\iint_S \frac{\partial}{\partial t}\mathbf{B}\cdot d\mathbf{S} \tag{2.2.14}$$

The normal of the surface element $d\mathbf{S}$ can be pointing in an arbitrary direction, and the surface S can be very small. Then the integral can be removed, and one has

$$\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{2.2.15}$$

[4] Faraday's law is experimentally motivated. Michael Faraday (1791-1867) was an extraordinary experimentalist who documented this law with meticulous care. It was only decades later that a mathematical description of this law was arrived at.

The above is Faraday's law in differential operator form.

In the static limit, $\frac{\partial \mathbf{B}}{\partial t} = 0$, giving

$$\nabla \times \mathbf{E} = 0 \tag{2.2.16}$$

2.2.2 Physical Meaning of Curl Operator

The curl operator $\nabla\times$ is a measure of the rotation or the circulation of a field at a point in space. On the other hand, $\oint_{\Delta C} \mathbf{E} \cdot d\mathbf{l}$ is a measure of the circulation of the field $\mathbf{E}$ around the loop formed by C. Again, the curl operator has its complicated representations in other coordinate systems like cylindrical or spherical, a subject that will not be discussed in detail here.

It is to be noted that our proof of the Stokes's theorem is for a flat open surface S, and not for a general curved open surface. Since all curved surfaces can be tessellated into a union of flat triangular surfaces according to the tiling theorem, the generalization of the above proof to curved surface is straightforward. An example of such a triangulation of a curved surface into a union of flat triangular surfaces is shown in Figure 2.7.

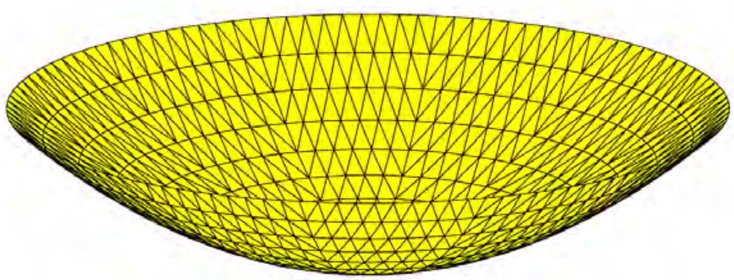

Figure 2.7: An arbitrary curved surface can be triangulated with flat triangular patches. The triangulation can be made arbitrarily accurate by making the patches arbitrarily small.

2.3 Maxwell's Equations in Differential Operator Form

With the use of Gauss' divergence theorem and Stokes' theorem, Maxwell's equations can be written more elegantly in differential operator forms. They are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.3.1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{2.3.2}$$

$$\nabla \cdot \mathbf{D} = \varrho \tag{2.3.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.3.4}$$

These equations are point-wise relations as they relate the left-hand side and right-hand sidefield values at a given point in space. Moreover, they are not independent of each other. For instance, one can take the divergence of the first equation (2.3.1), making use of the vector identity that $\nabla \cdot \nabla \times \mathbf{E} = 0$, one gets

$$-\frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = 0 \rightarrow \nabla \cdot \mathbf{B} = \text{constant} \tag{2.3.5}$$

This constant corresponds to magnetic charges, and since they have not been experimentally observed, one can set the constant to zero. Thus the fourth of Maxwell's equations, (2.3.4), follows from the first (2.3.1).

Similarly, by taking the divergence of the second equation (2.3.2), and making use of the current continuity equation that

$$\nabla \cdot \mathbf{J} + \frac{\partial \varrho}{\partial t} = 0 \tag{2.3.6}$$

one can obtain the second last equation (2.3.3). Notice that in (2.3.3), the charge density ϱ can be time-varying, whereas in the previous lecture, we have "derived" this equation from Coulomb's law using electrostatic theory.

The above logic follows if $\partial/\partial t \neq 0$, and is not valid for static case. In other words, for statics, the third and the fourth equations are not derivable from the first two. Hence all four Maxwell's equations are needed for static problems. For electrodynamic problems, only solving the first two suffices.

Something is amiss in the above. If $\mathbf{J}$ is known, then solving the first two equations implies solving for four vector unknowns, $\mathbf{E}, \mathbf{H}, \mathbf{B}, \mathbf{D}$, which has 12 scalar unknowns. But there are only two vector equations or 6 scalar equations in the first two equations. Thus we need more equations. These are provided by the constitutive relations that we shall discuss next.

2.4 Homework Examples

Example 1

If $\mathbf{D} = (2y^2 + z)\hat{x} + 4xy\hat{y} + x\hat{z}$, find:

1. Volume charge density ρ_v at $(-1, 0, 3)$.
2. Electric flux through the cube defined by

$$0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1.$$

3. Total charge enclosed by the cube.

Example 2

Suppose $\mathbf{E} = \hat{\mathbf{x}}3y + \hat{\mathbf{y}}x$, calculate $\int \mathbf{E} \cdot d\boldsymbol{l}$ along a straight line in the x-y plane joining (0,0) to (3,1).